

The Jacobsthal and Jacobsthal-Lucas sequences associated with pseudo graphs

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February 17, 2012

Abstract

In the present paper, we define two directed pseudo graph. Then, we investigate the adjacency matrices of the defined graphs and show that the permanents of the adjacency matrices are Jacobsthal and Jacobsthal-Lucas numbers. We also give complex factorization formulas for the Jacobsthal sequence.

1 Introduction

The *Jacobsthal* and *Jacobsthal-Lucas* sequences are defined by the following recurrence relations, respectively:

$$J_{n+2} = J_{n+1} + 2J_n \quad \text{where } J_0 = 0, J_1 = 1,$$

$$j_{n+2} = j_{n+1} + 2j_n, \quad \text{where } j_0 = 2, j_1 = 1,$$

for $n \geq 0$. The first few values of the sequences are given by the following table;

n	1	2	3	4	5	6	7	8	9
J_n	1	1	3	5	11	21	43	85	171
j_n	1	5	7	17	31	65	127	257	511

The *permanent* of a matrix is similar to the determinant but all of the signs used in the Laplace expansion of minors are positive. The permanent of an n -square matrix is defined by

$$\text{per} A = \sum_{\sigma \in S_n} \prod_{i=1}^n a_{i\sigma(i)}$$

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where the summation extends over all permutations σ of the symmetric group S_n [2].

Let $A = [a_{ij}]$ be an $m \times n$ matrix with row vectors r_1, r_2, \dots, r_m . We call A *contractible* on column k , if column k contains exactly two non zero elements. Suppose that A is contractible on column k with $a_{ik} \neq 0 \neq a_{jk}$ and $i \neq j$. Then the $(m-1) \times (n-1)$ matrix $A_{ij:k}$ obtained from A replacing row i with $a_{jk}r_i + a_{ik}r_j$ and deleting row j and column k is called the *contraction* of A on column k relative to rows i and j . If A is contractible on row k with $a_{ki} \neq 0 \neq a_{kj}$ and $i \neq j$, then the matrix $A_{k:ij} = [A_{ij:k}^T]^T$ is called the contraction of A on row k relative to columns i and j . We know that if A is a nonnegative matrix and B is a contraction of A [6], then

$$\text{per} A = \text{per} B. \quad (1)$$

A *directed pseudo graph* $G = (V, E)$, with set of vertices $V(G) = \{1, 2, \dots, n\}$ and set of edges $E(G) = \{e_1, e_2, \dots, e_m\}$, is a graph in which loops and multiple edges are allowed. A directed graph represented with arrows on its edges, each arrow pointing towards the head of the corresponding arc. The *adjacency matrix* $A(G) = [a_{i,j}]$ is $n \times n$ matrix, defined by the rows and the columns of $A(G)$ are indexed by $V(G)$, in which $a_{i,j}$ is the number of edges joining v_i and v_j [9].

It is known that there are a lot of relations between determinants or permanents of matrices and well-known number sequences. For example, in [4], the authors investigated the relations between Hessenberg matrices and Pell and Perrin numbers. In [5], the authors gave a relation between determinants of tridiagonal matrices and Lucas sequence. Secondly, they obtained a complex factorization formula for Lucas sequence. In [6], the authors considered the relationships between the sums of Fibonacci and Lucas numbers by Hessenberg matrices.

In [7], the author investigated general tridiagonal matrix determinants and permanents. Also he showed that the permanent of the tridiagonal matrix based on $\{a_i\}$, $\{b_i\}$, $\{c_i\}$ is equal to the determinant of the matrix based on $\{-a_i\}$, $\{b_i\}$, $\{c_i\}$.

In [8], the authors found $(0, 1, -1)$ tridiagonal matrices whose determinants and permanents are negatively subscripted Fibonacci and Lucas numbers. Also, they give an $n \times n$ $(1, -1)$ matrix S , such that $\text{per} A = \det(A \circ S)$, where $A \circ S$ denotes Hadamard product of A and S . Let S be a $(1, -1)$ matrix of order n , defined with

$$S = \begin{bmatrix} 1 & 1 & \cdots & 1 & 1 \\ -1 & 1 & \cdots & 1 & 1 \\ 1 & -1 & \cdots & 1 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & -1 & 1 \end{bmatrix}. \quad (2)$$

In [9], the authors investigate Jacobsthal numbers and obtain some properties for the Jacobsthal numbers. They also give Cassini-like formulas for Jacobsthal numbers as:

$$J_{n+1}J_{n-1} - J_n^2 = (-1)^n 2^{n-1} \quad (3)$$

In [10], the authors investigate incomplete Jacobsthal and Jacobsthal-Lucas numbers.

In [11], the author derived an explicit formula which corresponds to the Fibonacci numbers for the number of spanning trees given below:

In [12], the authors consider the number of independent sets in graphs with two elementary cycles. They described the extremal values of the number of independent sets using Fibonacci and Lucas numbers. In [13], the authors give a generalization for known-sequences and then they give the graph representations of the sequences. They generalize Fibonacci, Lucas, Pell and Tribonacci numbers and they show that the sequences are equal to the total number of k -independent sets of special graphs.

In [14], the authors present a combinatorial proof that the wheel W_n has $L_{2n} - 2$ spanning trees, L_n is the n th Lucas number and that the number of spanning trees of a related graph is a Fibonacci number.

In [15], the authors consider certain generalizations of the well-known Fibonacci and Lucas numbers, the generalized Fibonacci and Lucas p -numbers. Then they give relationships between the generalized Fibonacci p -numbers $F_p(n)$, and their sums, $\sum_{i=1}^n F_p(i)$, and the 1-factors of a class of bipartite graphs. Further they determine certain matrices whose permanents generate the Lucas p -numbers and their sums.

In [16], Lee considered k -Lucas and k -Fibonacci sequences and investigated the relationships between these sequences and 1-factors of a bipartite graph.

In the present paper, we investigate relationships between adjacency matrices of graphs and the Jacobsthal and Jacobsthal-Lucas sequences. We also give complex factorization formulas for the Jacobsthal numbers.

2 Determinantal representations of the Jacobsthal and Jacobsthal-Lucas numbers

In this section, we consider a class of pseudo graph given in Figure 1 and Figure 2, respectively. Then we investigate relationships between permanents of the adjacency matrices of the graphs and the Jacobsthal and Jacobsthal-Lucas numbers.

Figure 1

Let $H_n = [h_{ij}]_{n \times n}$ be the adjacency matrix of the graph given by Figure 1, in which the subdiagonal entries are 1s, the main diagonal entries are 1s, except the first one which is 3, the superdiagonal entries are 2s and otherwise 0. In other words:

$$H_n = \begin{bmatrix} 3 & 2 & & & & \\ 1 & 1 & 2 & & & 0 \\ & 1 & 1 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 2 \\ 0 & & & & 1 & 1 & 2 \\ & & & & & 1 & 1 \end{bmatrix} \quad (4)$$

Theorem 1 Let H_n be an n -square matrix as in (4), then

$$\text{per} H_n = \text{per} H_n^{(n-2)} = J_{n+2}$$

where J_n is the n th Jacobsthal number.

Proof. By definition of the matrix H_n , it can be contracted on column 1. Let $H_n^{(r)}$ be the r th contraction of H_n . If $r = 1$, then

$$H_n^{(1)} = \begin{bmatrix} 5 & 6 & & & & 0 \\ 1 & 1 & 2 & & & \\ & 1 & 1 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 2 \\ 0 & & & & 1 & 1 \end{bmatrix}.$$

Since $H_n^{(1)}$ also can be contracted according to the first column, we obtain;

$$H_n^{(2)} = \begin{bmatrix} 11 & 10 & & & & 0 \\ 1 & 1 & 2 & & & \\ & 1 & 1 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 2 \\ 0 & & & & 1 & 1 \end{bmatrix}.$$

Going with this process, we have

$$H_n^{(3)} = \begin{bmatrix} 21 & 22 & & & 0 \\ 1 & 1 & 2 & & \\ & 1 & 1 & 2 & \\ & & \ddots & \ddots & \ddots \\ 0 & & & 1 & 1 & 2 \\ & & & & 1 & 1 \end{bmatrix}.$$

Continuing this method, we obtain the r th contraction

$$H_n^{(r)} = \begin{bmatrix} J_{r+2} & 2J_{r+1} & & & 0 \\ 1 & 1 & 2 & & \\ & 1 & 1 & 2 & \\ & & \ddots & \ddots & \ddots \\ & & & 1 & 1 & 2 \\ 0 & & & & 1 & 1 \end{bmatrix}$$

where $2 \leq r \leq n-4$. Hence;

$$H_n^{(n-3)} = \begin{bmatrix} J_{n-1} & 2J_{n-2} & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

which, by contraction of $H_n^{(n-3)}$ on column 1,

$$H_n^{(n-2)} = \begin{bmatrix} J_n & 2J_{n-1} \\ 1 & 1 \end{bmatrix}.$$

By (1), we have $\text{per}H_n = \text{per}H_n^{(n-2)} = J_{n+2}$. ■

Let $K_n = [k_{ij}]_{n \times n}$ be the adjacency matrix of the pseudo graph given in Figure 2, with subdiagonal entries are 1s, the main diagonal entries are 1s, except the second one which is 3, the superdiagonal entries are 2s and otherwise 0. That is:

Figure 1:

Figure 2

and the adjacency matrix is:

$$K_n = \begin{bmatrix} 1 & 2 & & & & \\ 1 & 3 & 2 & & & 0 \\ & 1 & 1 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 2 \\ 0 & & & & 1 & 1 & 2 \\ & & & & & 1 & 1 \end{bmatrix} \quad (5)$$

Theorem 2 Let K_n be an n -square matrix as in (5), then

$$\text{per} K_n = \text{per} K_n^{(n-2)} = j_n$$

where j_n is the n th Jacobsthal-Lucas number.

Proof. By definition of the matrix K_n , it can be contracted on column 1. Namely,

$$K_n^{(1)} = \begin{bmatrix} 5 & 2 & & & & 0 \\ 1 & 1 & 2 & & & \\ 0 & 1 & 1 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 2 \\ 0 & & & & 1 & 1 \end{bmatrix}.$$

$K_n^{(1)}$ also can be contracted on the first column, so we get;

$$K_n^{(2)} = \begin{bmatrix} 7 & 10 & & & & 0 \\ 1 & 1 & 2 & & & \\ 0 & 1 & 1 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 2 \\ 0 & & & & 1 & 1 \end{bmatrix}.$$

Continuing this process, we have

$$K_n^{(r)} = \begin{bmatrix} j_{r+2} & 2j_{r+1} & & & & 0 \\ 1 & 1 & 2 & & & \\ 0 & 1 & 1 & 2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & 1 & 1 & 2 \\ 0 & & & & 1 & 1 \end{bmatrix}$$

for $1 \leq r \leq n-4$. Hence

$$K_n^{(n-3)} = \begin{bmatrix} j_{n-2} & 2j_{n-3} & 0 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

which by contraction of $K_n^{(n-3)}$ on column 1, gives

$$K_n^{(n-2)} = \begin{bmatrix} j_{n-1} & 2j_{n-2} \\ 1 & 1 \end{bmatrix}.$$

By applying (1) we have $\text{per}K_n = \text{per}K_n^{(n-2)} = j_n$, which is desired. ■

See Appendix B.

Let S be a matrix as in (2) and denote the matrices $H_n \circ S$ and $K_n \circ S$ by A_n and B_n , respectively. Thus

$$A_n = \begin{bmatrix} 3 & 2 & & & \\ -1 & 1 & 2 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1 & 2 \\ & 0 & & -1 & 1 & 2 \\ & & & & -1 & 1 \end{bmatrix} \quad (6)$$

and

$$B_n = \begin{bmatrix} 1 & 2 & & & \\ -1 & 3 & 2 & & 0 \\ & \ddots & \ddots & \ddots & \\ & & -1 & 1 & 2 \\ & 0 & & -1 & 1 & 2 \\ & & & & -1 & 1 \end{bmatrix}. \quad (7)$$

Then, we have

$$\det(A_n) = \text{per}H_n = J_{n+2}$$

and

$$\det(B_n) = \text{per}K_n = j_n.$$

In order to check the results, a Matlab and a Maple procedures are given in Appendix A and Appendix B for the matrices H_n and K_n respectively..

Appendix A. We give a Matlab source code to check permanents of the matrices given by (4).

```

clc; clear;
x = [];
n = ..;
x = eye(n);
x(1,1) = 3;
x(2,2) = 1;
for i = 1 : n - 1
    x(i + 1, i) = 1;
    x(i, i + 1) = 2;
end
for i = 1 : n - 1
    s1 = []; s2 = [];

```

```

for  $j = 1 : n + 1 - i$ 
  s1( $j$ ) =  $x(1, 1) * x(2, j)$ ;
  s2( $j$ ) =  $x(2, 1) * x(1, j)$ ;
end
s1
s2
st=s1+s2
if  $i \sim = n - 1$ 
   $xy = x(3 : n + 1 - i, 2 : n + 1 - i)$ ;
   $xy = [st(:, 2 : n + 1 - i); xy]$ ;
else
   $xy = st(2)$ ;
end
 $x = xy$ 
end
x

```

Appendix B. We also give a Maple source code to check permanents of the matrices given by (5).

```

restart:
> with(LinearAlgebra):
> permanent:=proc(n)
> local i,j,k,c,C;
> c:=(i,j)->piecewise(i=j+1,1,j=i+1,2,j=2 and i=2,3,i=j,1);
> C:=Matrix(n,n,c):
> for k from 0 to n-3 do
> print(k,C):
> for j from 2 to n-k do
> C[1,j]:=C[2,1]*C[1,j]+C[1,1]*C[2,j]:
> od:
> C:=DeleteRow(DeleteColumn(Matrix(n-k,n-k,C),1),2):
> od:
> print(k,eval(C)):
> end proc:with(LinearAlgebra):
> permanent( );

```

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